3. A TWO-STEP DESIGN PROCEDURE FOR TRANSONIC FLOW

3.1 Elliptic Continuation Principle and local supersonic flow fields

Let us recall the basic equations (8) in the working plane ζ_0 (9). Since the name "Hodograph plane" is usually associated with the plane defined by the velocity components u, v (3), we will use here the more general description "Rheograph" for planes like ζ_0 or ζ. With this name it is intended to relate to the applicability of the rheoelectric analogy for description of two-dimensional gas flow.

Equations (8) are elliptic in the subsonic half plane $(\upsilon < 0, \vartheta)$ and hyperbolic in the supersonic half plane $(\upsilon > 0, \vartheta)$. A transonic flow example with occurrence of mixed subsonic - supersonic flow, say, a local supersonic region embedded in subsonic flow, with smooth transition of the flow properties across the sonic line will, therefore, map into contacting regions E and H in Rheograph ζ_0 see Fig. 1 a, b. We wish to describe quantitatively a solution of system (8) representing such a flow.

For a subsonic flow example a boundary value problem might be formulated in the physical plane z as well as in the Rheograph ζ_0 by prescribing Neumann- or Dirichlet-conditions along a given boundary. For our transonic problem this would require the solution of a nonlinear equation (3) or (4) of mixed type in z, or solution of the mixed type linear system (8) in ζ_0.For the latter the boundary value problem in ζ_0 is not well posed[9]. Tricomi's boundary value problem[10] is the proper formulation in ζ_0 it is different from prescribing an arc Ψ = const in the supersonic part of the Rheograph ζ_0.

We propose a different way to formulate the problem in ζ_0.This is possible if we restrict ourself to obtain some solution with a <u>resulting</u> closed arc Ψ = const and not with a <u>prescribed</u> one.

First, we omit the change of sign in the first of equations (8). We take the negative sign for both half-planes $\upsilon > 0, \upsilon < 0$, thus having an elliptic system for the subsonic and the supersonic Rheograph ζ_0. We now define a boundary value problem for this linear, elliptic system, as sketched in Fig. 2a. It is well posed and we assume to have a method to obtain a solution. This solution will, locally, be one of the correct mixed type system (8) in region E_1 where $\upsilon < 0$, but it is a fictitious one in E_2 for $\upsilon > 0$, because real compressible flow requires solution of the hyperbolic part of (8) with the positive sign for $\upsilon > 0$. The solution in E_2 has here the purpose to provide a reasonable solution in E_1 with sonic line data

$$\begin{aligned} \phi^*(\vartheta) &= \phi(\upsilon = 0, \vartheta_A \ge \vartheta \ge \vartheta_B) \\ \Psi^*(\vartheta) &= \Psi(\upsilon = 0, \vartheta_A \ge \vartheta \ge \vartheta_B) \end{aligned} \quad (27)$$

This can be achieved also with some modification of the fictitious elliptic system in E_2: The coefficient $K(\upsilon)$ can be changed in some prescribed way, as long as it stays real and positive in E_2. One possibility is taking simply

$$K(\upsilon > 0) = K_E = const \quad (28)$$

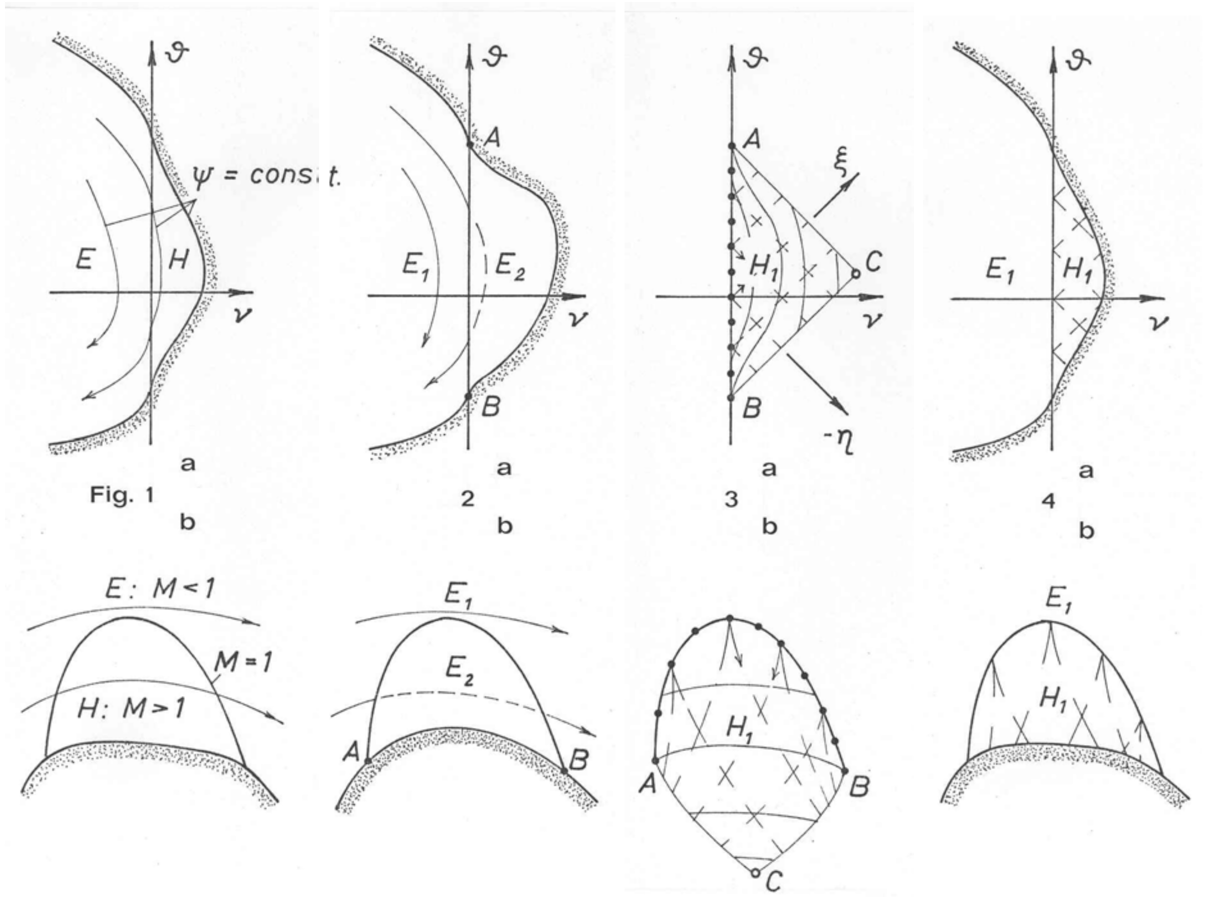

Figure 1-4. Elliptic Continuation Principle

which would result in a Cauchy-Riemann system in E_2. In this case, the part in E_2 of the elliptic solution can then easily be described analytically if the resulting data $\phi^*, \Psi^*(\vartheta)$ at $\upsilon = 0$ are expanded in terms of a harmonic analysis.

There is also a physical interpretation for this artificial solution in the flow plane x, y if υ in E_2 is reinterpreted

$$\upsilon(E_2) = \upsilon_E = \ln\frac{q_E}{a^*} \tag{29}$$

and K_E takes the value ρ_0/ρ^* :

The solution of the elliptic system in E_2 represents an example of "supersonic incompressible flow" with critical constant density ρ^*, embedded into the subsonic compressible solution obtained in E_1, see Fig. 2b. Streamlines, and most important, the streamline $\Psi = \Psi^*_A = \Psi^*_B = const$ defining our flow boundary for this fictitious flow, are integrated by use of (16) with the velocity variables q_E, ϑ, and $\rho = \rho^*$. The whole solution in $E_1 + E_2$ results in a flow with density obeying isentropic flow relations (2) only up to sonic velocity, beyond it density is frozen to the critical value. This interpretation led to a design method[11] which is not restricted to two-dimensional flow, results will be presented later.

We return now to our problem in the plane ζ_0. We still have to solve the equations for the real supersonic part of the flow, represented by the hyperbolic system (8) with positive sign and valid in the half plane $\zeta_0(\upsilon > 0)$. We choose the characteristic form of this system as outlined in (13) - (15). With the given data ϕ^*, Ψ^* along the ϑ-axis in the given interval AB we can solve this initial value problem at the sonic line with the method of characteristics. Although well known and used for many practical problems, we would like to stress the fact that we solve the system in the characteristic triangle ABC (Fig. 3 a) by calculating downstream along characteristics $\xi = const$, and upstream along characteristics $\eta = const$, with ξ, η defined in (13). Starting at AB we proceed toward C, the method therefore being a marching procedure normal to the flow direction, from the sonic line to a surface streamline yet to be determined. This concept is, in principle, also used in a procedure to calculate three-dimensional flow fields[12,13]. A line $\Psi = \Psi_A^* = \Psi_B = const$ is found in triangle ABC (it is different from the prescribed boundary in E_2!) and if it does not intersect one of the characteristics $\xi = const$. and $\eta = const$. more than once, then its integration (16) in the physical plane, see Fig. 3b, will give a new streamline arc AB and, along it, a velocity and pressure distribution. We use only the part between this streamline and the sonic line for our flow example and call this flow field H_1.

We go back now to our all-elliptic solution $E_1 + E_2$, Fig. 2b, and replace the part E_2 and also the surface streamline arc AB by the solution H_1 and its new arc AB of Fig. 3 b. This gives us a mixed subsonic - supersonic flow field which is a solution of the linear mixed system (8) in the hodograph plane, Fig. 4a, but also one of the nonlinear mixed system (3), or equations (4), in the physical plane, Fig. 4b. It can be shown, that the new arc AB of H_1 fits smoothly into the E_1 subsonic boundary, streamline curvature across any point on the sonic line is continuous.

We have outlined a method to obtain solutions for transonic flow, to be applied mainly to subsonic flows with embedded local supersonic regions. Applications to flows with predominantly supersonic flow and embedded subsonic regions involve the treatment of bow and tail shock waves, results have been obtained for airfoil flow with sonic or slightly supersonic freestream conditions only in special cases where analytical solutions of the near sonic equations (19) were applicable. An example will be illustrated later, to show transition from the problem of supercritical airfoil flow with subsonic freestream conditions, to sonic and slightly supersonic freestream conditions. However, supercritical flow is our main concern here, and more precisely, the use of the idea outlined for design of such flows which are shock-free.

3.2 The Rheograph structure of supercritical airfoil flow

The structure of supercritical airfoil flow is well known and needs no explanation here. However, some details are treated here shortly because they are of consequence for the practical indirect design method which will be outlined later.

We know from incompressible flow past lifting airfoils, that the isotachs in the flow field near the pressure (lower) surface exhibit a saddlepoint. This is the result of locally contracting streamlines due to the far field-effective circulation and the near field-effective body thickness. For compressible flow including supercritical conditions with or without a recompression shock, this is equally true, lines of constant local Mach number form a saddlepoint N below the lifting airfoil, see Fig. 5 a. This point is of interest for the mapping of a, say, given result of airfoil flow into our Rheograph plane ζ_0, because we want to know the principal structure of the boundary conditions for such flows in order to design new examples. The airfoil image in ζ_0 for shock-free flow shows two complications in view of formulating a closed elliptic boundary value probtem according to the first step of our design procedure:

First, the stagnation point of the airfoil is mapped into $\upsilon(M) = 0) = \infty$. Second, a part of the flowfield obviously covers the plane ζ_0, as indicated by the loop in the airfoil image. The structure of the field image has to be completed now with the mapping of the aforementioned saddlepoint N, defined by $\upsilon(M_N), \vartheta_N$. A second Riemann sheet provides the second deck of ζ_0, it is, connected with the basic deck along a cut from the airfoil mapping intersection to the point N, forming a branchpoint in the Rheograph ζ_0.

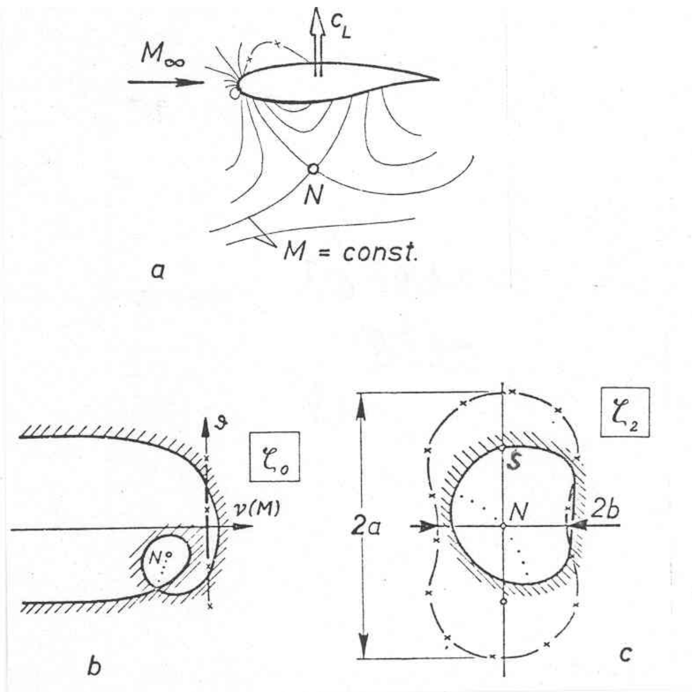

Figure 5. Saddlepoint, lift coefficient and Rheographs ζ_0, ζ_2

A detailed description of the mathematical structure of these flow properties has been given in[14]. In order to arrive at a single-sheeted boundary value problem of closed, finite structure we perform now two mappings (9): first the stagnation point S is moved into a finite domain with the mapping

$$\zeta_1 = e^{\zeta_0} \tag{30}$$

Another mapping unwraps the loop of the airfoil image and we obtain a single sheeted domain by

$$\begin{aligned} \zeta_2 &= c(\zeta_1 - \zeta_{1N})^{1/2} \\ \zeta_{1N} &= e^{\upsilon(M_N) + i\vartheta_N} \end{aligned} \tag{31}$$

with c an arbitraty scaling constant. The airfoil image maps in ζ_2 into a closed curve including the stagnation point S as illustrated in Fig. 5 c. The aforementioned saddlepoint maps into the origin, the sonic line into a Cassini curve or outer lemniscate with half axes b/a. The ratio b/a is a function of the local Mach number M_N in the saddlepoint:

$$b/a = \sqrt{\frac{1 - e^{\upsilon(M_N)}}{1 + e^{\upsilon(M_N)}}} \tag{32}$$

The value M_N is related to the Mach number at infinity M_∞ in a similar way as the velocities in the saddlepoint and at infinity for an incompressible flow example past a Joukowsky airfoil or a circular cylinder with circulation. These latter examples are known analytically and from these we arrive at the ratio M_N/M_∞ as a function of the lift coefficient C_L

$$M_N / M_\infty \sim 1 - A c_L^2 \tag{33}$$

with a constant A. The circular cylinder example gives at least an idea about the magnitude of A:

$$A \sim 1/(2\pi^2) \tag{34}$$

These relations invite to be checked on airfoil flow examples. We have a possibility to do this with existing results for hodograph supercritical airfoil design examples by Nieuwland[15], Boerstoel[14], or by Garabedian and Korn[16]. Some of these authors' designs are evaluated in Fig. 6, we see that the given relations (33), (34) are fulfilled satisfactorily for not too large c_L. We conclude that for given (b/a) in (32) and for given M_∞ obviously a certain band of c_L is possible. We stress this fact because we will later use an electric analog flow tool which will work with devices designed for fixed b/a where the given relations and the diagram Fig. 6 provides possible lift coefficients $c_L(M_\infty, b/a)$.

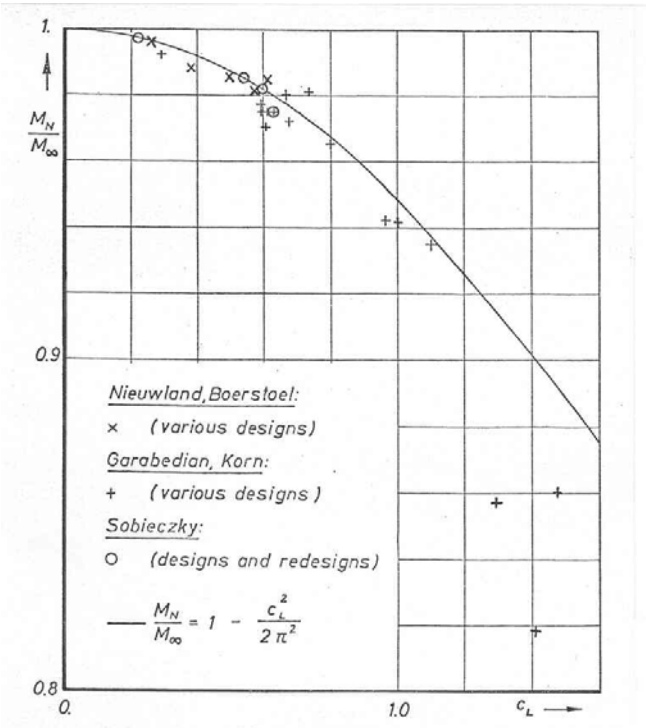

Figure 6. Mach number ratio vs. lift coefficient

3.3 Free stream singularities in the Rheograph plane

We further investigate the structure of supercritical flow in our working plane ζ_2. With the mapping of the airfoil into a closed curve as sketched in Fig. 5c. the domain enclosed is the mapping of the whole flow. Infinity in the physical plane with $M = M_\infty, \vartheta = 0$ maps into a point I, where the solution of system (10) has a singularity. It has the structure

$$\phi - iK_I\Psi_I = A(\zeta_2 - \zeta_{2I})^{-1} + B\ln(\zeta_2 - \zeta_{2I}) \tag{35}$$

The first term is a dipole, with the axis defined by the complex coefficient A. The second term is representing circulation, with B an imaginary coefficient. For nonlifting flow B vanishes and, in the case of a symmetrical airfoil, I and N coincide, $M_N = M_\infty$, the airfoil mapping is symmetrical to the vertical axis of ζ_2. For lifting airfoil flow the free stream singularity I is situated between saddlepoint N and the sonic line M = 1, see Fig. 7 a. For higher subsonic Mach numbers M_∞, I moves toward the sonic line and, with c_L fixed, M_N therefore has to be higher too. This results in a smaller "waist" b/a(32), as sketched in Fig. 7 b. Finally, arriving at sonic freestream conditions, the waist reduces to zero, Fig. 7c. This limiting case of airfoil flow with $M_\infty = 1$ is already beyond the relations (32) -(35) for supercritical conditions. Nevertheless, it is an interesting topic to study the change of hodograph structures if $M_\infty \to 1$, arriving from $M_\infty < 1$.

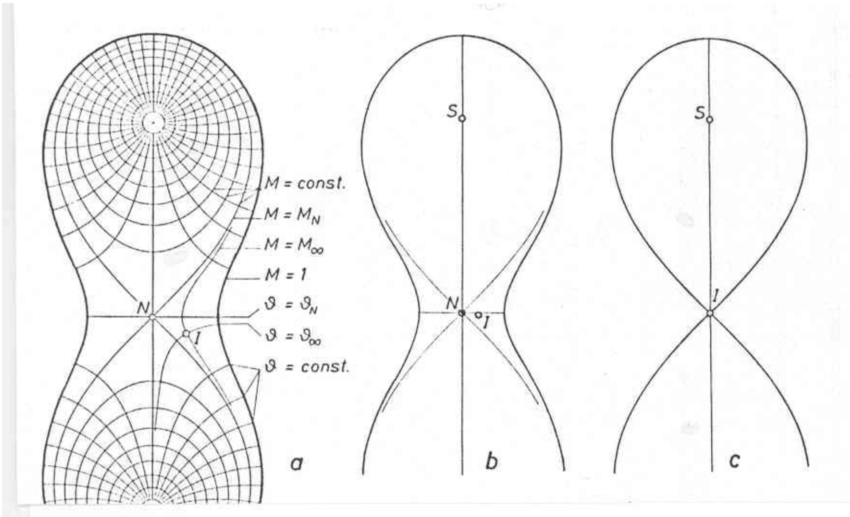

Figure 7. Rheograph ζ_2 for freestream conditions $M_\infty \to 1$

Airfoils with round leading edge have a stagnation point, which results in the fact, that the airfoil image in ζ_2 includes the mapped stagnation point S , see Fig. 5c, or for sonic free stream conditions, Fig. 8a. There are analytical results of the near sonic equations (19) for cusped airfoils[3,5,17] in sonic flow, with a sharp leading edge in smooth entry conditions therefore having no stagnation point. The airfoil contour wetted by subsonic flow maps into a region around the free stream singularity in I*, see Fig. 8b. This singularity is different from the subsonic far field solution (35), the transition from one to the other involves far field influence of the tail shock wave, similar to the transition from $M_\infty > 1$ to sonic free stream involving the far field of a detached bow wave. The latter problem is solved analytically[18] with use of the transonic shock polar mapped into the near sonic Rheograph Fig. 8c.

We give some detailed illustrations for the aforementioned analytical results of cusped airfoil flow in Figures 9 -11 although their value for practical flows is limited. On the other hand, these results represent educational examples for transonic flow phenomena, where the problem is solved for the subsonic part first, with the supersonic part either given analytically together with the subsonic results, or being calculated starting at suitable initial conditions provided by the subsonic solution.

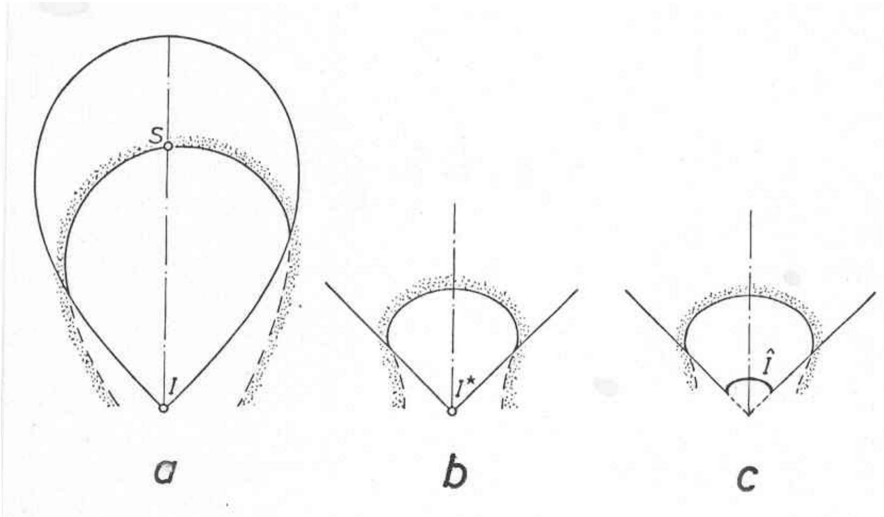

Figure 8. Rheograph ζ_2 for freestream conditions $M_\infty \geq 1$

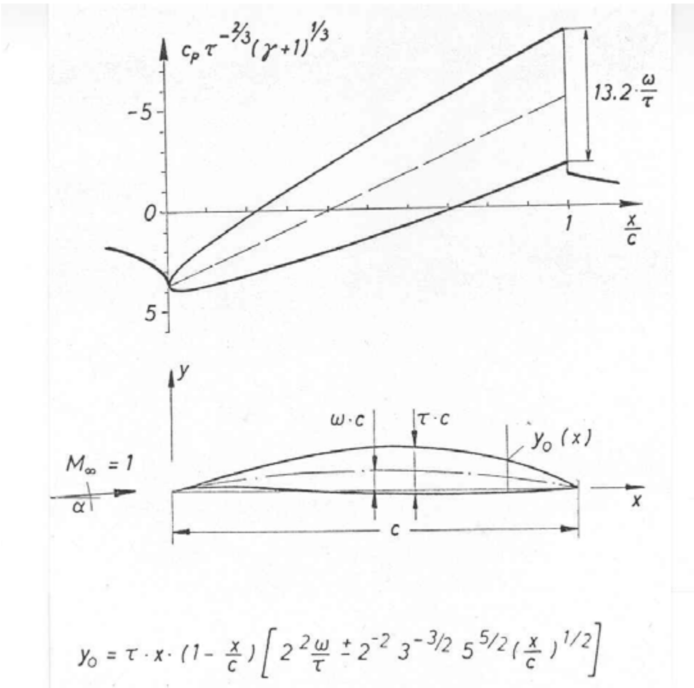

Figure 9. Cusped lifting airfoil in sonic freestream

In Fig. 9 the cusped airfoil and its geometry formula is drawn. The sonic free-stream $M_\infty = 1$ has a certain angle of attack, α, which leads to smooth entry conditions, with α, but also the local pressure on the airfoil, lift and drag, functions of the camber/thickness ratio ω / τ, see Fig. 10 a, b. This analytical result is a generalization of Guderley's cusp, where $\omega/\tau = 0$, a detailed description is given in[17].

There are results also for supersonic Mach numbers. Fig. 11a shows a configuration of a bow wave and the local subsonic far field in a similarity flow plane, which illustrates the extent of a local subsonic region for $M_\infty \to 1$.

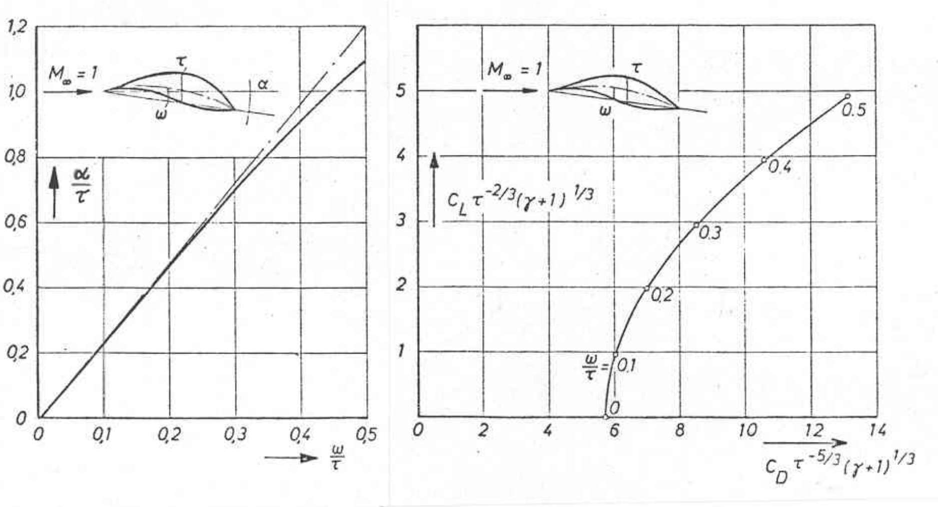

Figure 10. Cusped lifting airfoil: Smooth entry conditions, lift, drag

In terms of airfoil geometry, this extent is illustrated in Fig. 11b for a Guderley cusp. Stand - off distance of a detached bow wave is obtained, e.g. for a 10% thick airfoil the bow wave attaches at $M_\infty = 1,17$. The extent of the local subsonic field normal to the flow direction is large, as the far field solution Fig. 11a indicates. This is important for wind tunnel tests with detached bow waves, where the tunnel wall should not be reached by the subsonic field. For a 10 % thick Guderley - airfoil, at $M_\infty = 1,15$ for instance, the distance from the airfoil to the ends of the subsonic field is about four times the chord length, while the bow wave stand off distance from the cusp is only a fifth of the chord length.

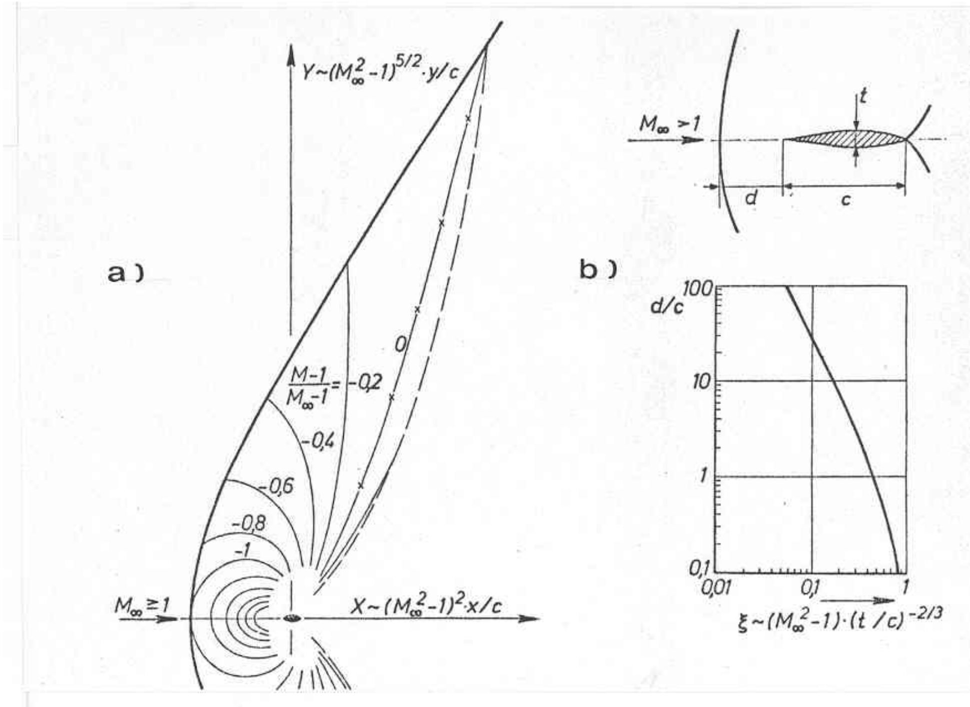

Figure 11. Detached bow wave: a) Similarity solution for $M_\infty \to 1$
b) Cusped airfoil stand-off distance